Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

deduced from one in the paper of Professor Boole on the Theory of Development:

$$f_{n+1}(x)=\frac{x}{n+1}f'_n(x)-\frac{n}{n+1}f_n(x).$$

The method of the present paper is of course of far more general application; but I have said enough in it to explain the principle on which such expansions must be conducted.

IV. "On the Summation of Series." By W. H. L. Russell, Esq., A.B. Communicated by Professor Stokes, Sec. R.S. Received May 13, 1865.

In a Memoir published in the Philosophical Transactions for the year 1855, I applied the Theory of Definite Integrals to the summation of many intricate series. I have thought my researches on this subject might well be terminated by the following paper, in which I have pointed out methods for the summation of series of a far more complicated nature.

I commence with some remarks intended to give clear conceptions of the general method of calculation.

In any series,

$$u_0+\alpha u_1+\alpha^2u_2+\alpha^3u_3+\&c.+\alpha^x u_x+\&c.$$

Where α is less than unity, it is evident that we can sum the series by a definite integral when $u_x=\int du\, U_1\, U^x$, U_1 and U being functions of u, and the integral being taken between certain assigned limits. For it is manifest that the quantity under the integral sign then becomes a geometrical progression.

Again, for a similar reason we can express by a definite integral the sum of the series

$$u_0v_0w_0\ldots+\alpha u_1v_1w_1\ldots+\alpha^2u_2v_2w_2\ldots+\&c.$$
$$+\alpha^x u_xv_xw_x\ldots+\&c.,$$

where

$$u_x=\int du\, U_1U^x,\quad v_x=\int dv V_1V^x,$$

$$w_x=\int dw W_1W^x,\ \&c.$$

Lastly, we can sum the series

$$u_0v_0w_0\ldots+\alpha\, u_1v_1w_1\ldots+\alpha^2u_2v_2w_2\ldots+\&c.$$
$$+\alpha^x u_xv_xw_x\ldots+\&c.$$

by a definite integral when

$$u_x = \int du\,\mathrm{U}_1\mathrm{U}^x + \int du'\,\mathrm{U}'_1\mathrm{U}'^x + \int du''\,\mathrm{U}''_1\mathrm{U}''^x + \ldots$$

$$v_x = \int dv\,\mathrm{V}_1\mathrm{V}^x + \int dv'\,\mathrm{V}'_1\mathrm{V}'^x + \int dv''\,\mathrm{V}''_1\mathrm{V}''^x + \ldots$$

$$w_x = \int dw\,\mathrm{W}_1\mathrm{W}^x + \int dw'\,\mathrm{W}'_1\mathrm{W}'^x + \int dw''\,\mathrm{W}''_1\mathrm{W}''^x + \ldots$$

$$\&\mathrm{c.} = \&\mathrm{c.},$$

the number of each set of quantities u_1, u', &c., v_1, v', &c., w_1, w', &c. being of course finite.

I shall now consider the series

$$\phi(0)^{\psi(0)} + \alpha\phi(1)^{\psi(1)} + \alpha^2\phi(2)^{\psi(2)} + \&\mathrm{c.}$$
$$+ \alpha^x\phi(x)^{\psi(x)} + \&\mathrm{c.},$$

where $\phi(x)$ and $\psi(x)$ are rational functions of (x).

Let

$$\phi(x) = \frac{(\alpha+\beta x)(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 x)\ldots}{(a+bx)(a_1+b_1 x)(a_2+b_2 x)\ldots},$$

$$\psi(x) = \frac{1}{m+nx} + \frac{1}{m_1+n_1 x} + \frac{1}{m_2+n_2 x} + \&\mathrm{c.}$$

Hence, by what has been said, the problem is reduced to finding a definite integral of the form $\int du\,\mathrm{U}_1\mathrm{U}_x$ equivalent to $\left\{\frac{\alpha+\beta x}{a+bx}\right\}^{\frac{1}{m+nx}}$.

Now

$$\left\{\frac{\alpha+\beta x}{a+bx}\right\}^{\frac{1}{m+nx}} = \frac{1}{\Gamma\left\{\frac{1}{m+nx}\right\}}\int_0^1 du\,.\,u^{\frac{a+bx}{\alpha+\beta x}-1}\left(\log_\epsilon\frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$= \frac{1}{\pi}\sin\frac{\pi}{m+nx}\,\Gamma\left\{\frac{m+nx-1}{m+nx}\right\}\int_0^1 du\,.\,u^{\frac{a+bx}{\alpha+\beta x}-1}\left(\log_e\frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$\sin\frac{\pi}{m+nx} = \frac{1}{\sqrt{2}}\cos\left\{\frac{\pi}{4}-\frac{\pi}{m+nx}\right\} - \frac{1}{\sqrt{2}}\sin\left\{\frac{\pi}{4}-\frac{\pi}{m+nx}\right\}$$

$$= \frac{\sqrt{2}}{\pi}\int_0^1 ds\,\log_\epsilon^{-\frac{1}{2}}\frac{1}{s}\,.\,s^{\frac{1}{m+nx}-1}\int_0^\infty dz\cos(m+nx)z^2\cos 2\sqrt{\pi}\,.\,z$$

$$- \frac{\sqrt{2}}{\pi}\int_0^1 ds\,\log_\epsilon^{-\frac{1}{2}}\frac{1}{s}\,.\,s^{\frac{1}{m+nx}-1}\int_0^\infty dz\sin(m+nx)z^2\cos 2\sqrt{\pi}\,.\,z$$

$$\Gamma\left\{\frac{m+nx-1}{m+nx}\right\}=\int_0^\infty \epsilon^{-v}\,dv\,\epsilon^{-\frac{1}{m+nx}}$$

$$\int_0^1 du\,.\,u^{\frac{a+bx}{\alpha+\beta x}-1}\left(\log_\epsilon\frac{1}{u}\right)^{\frac{1}{m+nx}-1}$$

$$=\int_{\frac{1}{\epsilon}}^1 du\,.\,\epsilon^{\log_\epsilon u\left(\frac{a+bx}{\alpha+\beta x}-1\right)}\,\epsilon^{\log_\epsilon\log_\epsilon\frac{1}{u}\left(\frac{1}{m+nx}-1\right)}$$

(where $\log_\epsilon\log_\epsilon\frac{1}{u}$ is negative),

$$+\int_0^{\frac{1}{\epsilon}} du\;\epsilon^{\log_\epsilon u\left(\frac{a+bx}{\alpha+\beta x}-1\right)}\,\epsilon^{\log_\epsilon\log_\epsilon\frac{1}{u}\left(\frac{1}{m+nx}-1\right)}$$

(where $\log_\epsilon\log_\epsilon\frac{1}{u}$ is positive).

By means of these transformations the series is reduced to forms considered in my previous investigations; for the general term of the transformed series included under the signs of definite integration is of the form

$$\mathrm{P}\alpha^x\,\epsilon^{\frac{\phi}{m+nx}}\{\cos h\,(m+nx)-\sin k\,(m+nx)\},$$

a form which I have discussed in the memoir in the Philosophical Transactions mentioned at the beginning of this paper.

Next let us investigate the series

$$\sqrt[n]{\psi(0)+\sqrt[m]{\chi(0)}}+\alpha\sqrt[n]{\psi(1)+\sqrt[m]{\chi(1)}}+\ldots+\alpha^x\sqrt[n]{\psi(x)+\sqrt{\chi(x)}}+\ldots,$$

where $\psi(x)$ and $\chi(x)$ are identical functions of (x).

We transform as follows:—

$$\sqrt[n]{\psi(x)+\sqrt[m]{\chi(x)}}=\frac{1}{\Gamma\left(\frac{1}{n}\right)}\int_0^1 du\,u^{\frac{1}{\psi(x)+\sqrt[m]{\chi(x)}}-1}\left(\log_\epsilon\frac{1}{u}\right)^{n-1}$$

$$\epsilon^{\frac{\log_\epsilon u}{\psi(x)+\sqrt[m]{\chi(x)}}}=\frac{\sqrt{\psi(x)+\sqrt[m]{\chi(x)}}}{2\sqrt{\pi}\log_\epsilon^{\frac{1}{2}}\frac{1}{u}}\int_{-\infty}^{\infty}d\rho\;\epsilon^{-\frac{\psi(x)+\sqrt[m]{\chi(x)}}{4\log_\epsilon\frac{1}{u}}\rho^2}\cos\rho,$$

remembering that $\log_\epsilon u$ is negative,

$$\sqrt{\psi(x)+\sqrt[m]{\chi(x)}}=\frac{\psi(x)+\sqrt[m]{\chi(x)}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\epsilon^{-v^2(\psi(x)+\sqrt[m]{\chi(x)})}dv,$$

$$\epsilon^{-\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\sqrt[m]{\chi(x)}}=\frac{1}{\pi}\int_0^\pi\frac{(1-\sqrt[m]{\chi(x)}^2)\mathrm{F}(\theta)d\theta}{1-2\sqrt[m]{\chi(x)}\cos\theta+\sqrt[m]{\chi(x)}^2},$$

where

$$\mathrm{F}(\theta)=\epsilon^{-\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\cos\theta}\cos\left\{\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\sin\theta\right\},$$

and $\sqrt[m]{\chi(x)}$ is supposed less than unity.

Hence also

$$\epsilon^{-\left(v^2+\frac{\rho^2}{4\log_\epsilon\frac{1}{u}}\right)\sqrt[m]{\chi(x)}}$$

$$=\frac{1}{\pi}\int_0^\pi\frac{(1-\sqrt[m]{\chi(x)^2})\mathrm{F}_1(\chi(x),\epsilon^{i\theta})\mathrm{F}_1(\chi(x),\epsilon^{-i\theta})\mathrm{F}(\theta)d\theta}{1-2\chi(x)\cos m\,\theta+\chi(x)^2},$$

where we render the denominator rational by multiplication, and suppose

$$\mathrm{F}_1(\chi(x),\epsilon^{i\theta})=\chi(x)^{\frac{m-1}{m}}+\epsilon^{i\theta}\,\chi(x)^{\frac{m-2}{m}}+\epsilon^{2i\theta}\chi(x)^{\frac{m-3}{m}}+\dots,$$

$$\mathrm{F}_1(\chi(x),\epsilon^{-i\theta})=\chi(x)^{\frac{m-1}{m}}+\epsilon^{-i\theta}\chi(x)^{\frac{m-2}{m}}+\epsilon^{-2i\theta}\chi(x)^{\frac{m-3}{m}}+\dots,$$

$$\frac{1}{1-2\chi(x)\cos m\theta+\chi(x)^2}=\frac{\chi_1(x)^2}{(\chi_1(x)-\cos m\theta)^2+\sin^2 m\theta},$$

where

$$\chi_1(x)=\frac{1}{\chi(x)}.$$

$\chi(x)$ is less than unity, hence $\chi_1(x)$ is greater than unity, and therefore $\chi_1(x)-\cos m\,\theta$ is always positive; hence

$$\frac{1}{1-2\chi(x)\cos m\,\theta+\chi(x)^2}=\frac{\chi(x)^2}{\sin m\,\theta}\int_0^\infty dz\epsilon^{-(\chi_1(x)-\cos m\theta)z}\sin(z\sin m\theta).$$

The general term of the series included under the signs of definite integration is now of the form

$$\mathrm{P}\psi(x)\epsilon^{\alpha\psi(x)}\epsilon^{-\frac{\beta}{\chi(x)}}\chi(x)^{\frac{r}{m}}\alpha^x,$$

belonging to a class which I have considered in my former memoir.

Let us now consider the series

$$\sqrt[n]{\phi(0)}^{\sqrt[m]{\psi(0)}}+\alpha.\sqrt[n]{\phi(1)}^{\sqrt[m]{\psi(1)}}+\alpha^2.\sqrt[n]{\phi(2)}^{\sqrt[m]{\psi(2)}}+\dots$$

$$+\alpha x.\sqrt[n]{\phi(x)}^{\sqrt[m]{\psi(x)}}+\&c.,$$

$\phi(x)$ and $\psi(x)$ being rational functions of (x).

$$\sqrt[n]{\phi(x)}^{\sqrt[m]{\psi(x)}}=\frac{1}{\Gamma\sqrt[m]{\psi(x)}}\int_0^1 u^{\frac{1}{\sqrt[n]{\phi(x)}}-1}\left(\log_\epsilon\frac{1}{u}\right)^{\sqrt[m]{\psi(x)}-1}du,$$

where $\psi(x)$ must be supposed less than unity, in order that the following transformation may hold :—

$$\frac{1}{\Gamma\sqrt[m]{\psi(x)}}=\frac{\sin\pi\sqrt[m]{\psi(x)}}{\pi}\Gamma(1-\sqrt[m]{\psi x})$$

$$\sin\pi\sqrt[m]{\psi(x)}=\frac{1}{2\pi}\int_0^\pi\frac{d\theta(1-\sqrt[m]{\psi(x)^2})d\theta(\sin\pi\epsilon^{i\theta}+\sin\pi\epsilon^{-i\theta})}{1-2\cos\theta\sqrt[m]{\psi(x)}+\sqrt[m]{\psi(x)^2}}$$

$$=\frac{1}{2\pi}\int_0^\pi\frac{d\theta(1-\sqrt[m]{\psi(x)^2})\mathrm{F}(\psi x\,\epsilon^{i\theta})\mathrm{F}(\psi(x),\epsilon^{-i\theta})\,(\sin\pi\epsilon^{i\theta}+\sin\pi\epsilon^{-i\theta})}{1-2\cos m\theta,\ \psi(x)+\psi(x)^2},$$

where

$$\mathrm{F}(\psi(x),\epsilon^{i\theta})\ =\psi(x)^{\frac{m-1}{m}}+\epsilon^{i\theta}\psi(x)^{\frac{m-2}{m}}+\ldots.,$$

$$\mathrm{F}(\psi(x),\epsilon^{-i\theta})=\psi(x)^{\frac{m-1}{m}}+\epsilon^{-i\theta}\psi(x)^{\frac{m-2}{m}}+\ldots.;$$

also

$$\Gamma(1-\sqrt[m]{\psi(x)}=\int_0^\infty\epsilon^{-v}v^{-\sqrt[m]{(\psi x)}}dv.$$

The remainder of the process will be evident from the two former examples.

V. “On a Theorem concerning Discriminants.” By J. J. SYLVESTER, F.R.S. Received May 27, 1865.

Let $\mathrm{F}(a, b, c, d)=a^2d^2+4a^3c+4d^3b-3a^2b^2-6abcd$, and let a, b, c, d be four quantities all greater than zero, which make this function vanish.

(1) The cubic equation in x, $\mathrm{F}(a, x, c, d)$ will have two positive roots (b, b_1); so $\mathrm{F}(a, b_1, x, d)$ will have two such roots (c, c_1), $\mathrm{F}(a, x, c_1, d)$ two such (b_1, b_2), $\mathrm{F}(a, b_2, x, d)$ two such (c_1, c_2), and so on *ad infinitum*; we may thus generate the infinite series $b_1\,c_1\,b_2\,c_2$ . . . .

Similarly, beginning with the equation $\mathrm{F}(a, b, x, d)$, and proceeding as above, we shall obtain a similar series, c', b', c'', b'' . . .; and combining the two together, and with the initial quantities b, c, we obtain a series proceeding to infinity in both directions . . . . . $b''\,c''\,b'\,c'\,b\,c\,b_1\,c_1\,b_2\,c_2$ . . . . .

(2) The four quantities

$$\frac{\delta\mathrm{F}}{\delta a},\ \frac{\delta\mathrm{F}}{\delta b},\ \frac{\delta\mathrm{F}}{\delta c},\ \frac{\delta\mathrm{F}}{\delta d},$$

where F represents $\mathrm{F}(a, b, c, d)$, will present one or the other of the three following successions of sign,

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ 0 & 0 & 0 & 0 \end{array}$$

(3) When the last is the case, *i.e.* when the differential derivatives all